

Asymptotic behaviour of solutions of the fast diffusion equation near its extinction time

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Abstract

Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\beta \geq \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. For any $\lambda > 0$, we will prove the existence and uniqueness (for $\beta \geq \frac{\rho_1}{n-2-nm}$) of radially symmetric singular solution $g_\lambda \in C^\infty(\mathbb{R}^n \setminus \{0\})$ of the elliptic equation $\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n \setminus \{0\}$, satisfying $\lim_{|x| \rightarrow 0} |x|^{\alpha/\beta} g_\lambda(x) = \lambda^{-\frac{\rho_1}{(1-m)\beta}}$. When β is sufficiently large, we prove the higher order asymptotic behaviour of radially symmetric solutions of the above elliptic equation as $|x| \rightarrow \infty$. We also obtain an inversion formula for the radially symmetric solution of the above equation. As a consequence we will prove the extinction behaviour of the solution u of the fast diffusion equation $u_t = \Delta u^m$ in $\mathbb{R}^n \times (0, T)$ near the extinction time $T > 0$.

Key words: extinction behaviour, fast diffusion equation, self-similar solution, higher order asymptotic behaviour

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1 Introduction

The equation

$$u_t = \Delta u^m \tag{1.1}$$

appears in many physical models. When $m > 1$, (1.1) is the porous medium equation which models the flow of gases or liquid through porous media. When $m = 1$, (1.1) is the heat equation. When $0 < m < 1$, (1.1) is the fast diffusion equation. When $m = \frac{n-2}{n+2}$, $n \geq 3$, and $g = u^{\frac{4}{n+2}} dx^2$ is a metric on \mathbb{R}^n which evolves by the Yamabe flow,

$$\frac{\partial g}{\partial t} = -Rg$$

where R is the scalar curvature of the metric g , then u satisfies [DKS], [PS],

$$u_t = \frac{n-2}{m} \Delta u^m.$$

It is because of the importance of the equation (1.1) and its relation to the Yamabe flow, there are a lot of research on this equation recently by P. Daskalopoulos, J. King, M. del Pino, N. Sesum, M. Sáez, [DKS], [DPS], [DS1], [DS2], [PS], S.Y. Hsu [Hs1–3], K.M. Hui [Hu1], [Hu2], M. Fila, J.L. Vazquez, M. Winkler, E. Yanagida, [FVWY], [FW], A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J.L. Vazquez, [BBDGV], [BDGV], etc. We refer the reader to the survey paper [A] by D.G. Aronson and the books [DK], [V2], by P. Daskalopoulos, C.E. Kenig, and J.L. Vazquez on the recent progress on this equation.

As observed by J.L. Vazquez [V1], M.A. Herrero and M. Pierre [HP], and others [Hs2], [Hu1], there is a big difference on the behaviour of solution of (1.1) for the case $\frac{n-2}{n} < m < 1$, $n \geq 3$, and the case $0 < m \leq \frac{n-2}{n}$, $n \geq 3$. For example for any $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, when $\frac{n-2}{n} < m < 1$, $n \geq 3$, there exists a unique global positive smooth solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$ with initial value u_0 [HP]. On the other hand when $0 < m < \frac{n-2}{n}$, $n \geq 3$, there exists $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, and $T > 0$ such that the solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \quad (1.2)$$

extincts at time T [DS1]. Since the asymptotic behaviour of the solution of (1.2) near the extinction time is usually similar to the asymptotic behaviour of the self-similar solution of (1.1), in order to understand the behaviour of the solution of (1.2) near the extinction time we will first study various properties of the self-similar solutions of (1.1) in this paper.

Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\beta > \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. For any $\lambda > 0$, by Theorem 1.1 of [Hs1] there exists a unique radially symmetric solution v_λ of the equation

$$\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, v > 0, \quad (1.3)$$

in \mathbb{R}^n that satisfies $v_\lambda(0) = \lambda$. By [Hs3], v_λ satisfies

$$\lim_{r \rightarrow \infty} r^2 v_\lambda(r)^{1-m} = \frac{2m(n-2-nm)}{(1-m)\rho_1}. \quad (1.4)$$

Note that when $\rho_1 = 1$, the function

$$\psi_\lambda(x, t) = (T-t)^\alpha v_\lambda((T-t)^\beta x) \quad (1.5)$$

is a solution of (1.1) in $\mathbb{R}^n \times (0, T)$ for any $T > 0$. On the other hand if $\rho_1 = 1$, $m = \frac{n-2}{n+2}$, and $n \geq 3$, then the metric

$$g = \left(\left(\frac{n-1}{m} \right)^{\frac{1}{1-m}} v_\lambda \right)^{\frac{4}{n+2}} dx^2 \quad (1.6)$$

on \mathbb{R}^n is a Yamabe shrinking soliton [DS2]. Conversely as proved by P. Daskalopoulos and N. Sesum [DS2] any Yamabe shrinking soliton on complete locally conformally flat manifold is of the form (1.6) where v_λ is a solution of (1.3) in \mathbb{R}^n for some $\alpha = \frac{2\beta+1}{1-m}$ with $v_\lambda(0) = \lambda$ for some constant $\lambda > 0$.

Let $\beta_1 = \frac{\rho_1}{n-2-nm}$,

$$\beta_0 = \begin{cases} \rho_1 \sqrt{\frac{2(1-m)}{n-2-nm}} & \text{if } 0 < m \leq \frac{n-2}{n+2} \\ \rho_1 \max \left(2 \sqrt{\frac{2(1-m)}{n-2-nm}}, \frac{(n+2)m - (n-2)}{n-2-nm} \right) & \text{if } \frac{n-2}{n+2} < m < \frac{n-2}{n}, \end{cases}$$

and γ_2, γ_1 , be the two roots of the equation

$$\gamma^2 - \left(\frac{n-2-(n+2)m}{1-m} + \frac{2\beta(n-2-nm)}{(1-m)\rho_1} \right) \gamma + \frac{2(n-2-nm)}{(1-m)} = 0 \quad (1.7)$$

given by

$$\gamma_i = \frac{1}{2(1-m)} \left\{ A(\beta) + (-1)^i \sqrt{A(\beta)^2 - 8(n-2-nm)(1-m)} \right\}, \quad i = 1, 2 \quad (1.8)$$

where

$$A(\beta) = n-2 - (n+2)m + \frac{2\beta(n-2-nm)}{\rho_1}.$$

Now if $\beta \geq \beta_0$, then

$$\begin{aligned} & A(\beta)^2 - 8(n-2-nm)(1-m) \\ & \geq \begin{cases} \frac{4(n-2-nm)^2}{\rho_1^2} \left\{ \beta^2 - \frac{2(1-m)\rho_1^2}{n-2-nm} \right\} & \text{if } 0 < m \leq \frac{n-2}{n+2} \\ \frac{(n-2-nm)^2}{\rho_1^2} \left\{ \beta^2 - \frac{8(1-m)\rho_1^2}{n-2-nm} \right\} & \text{if } \frac{n-2}{n+2} < m < \frac{n-2}{n} \end{cases} \\ & \geq 0. \end{aligned}$$

Hence $\gamma_2 \geq \gamma_1 > 0$ are real roots of (1.7) when $0 < m < \frac{n-2}{n}$, $n \geq 3$, and $\beta \geq \beta_0$. Note that

$$\beta > \beta_1 \quad (\beta = \beta_1) \Leftrightarrow \quad n\beta > \alpha \quad (n\beta = \alpha \text{ respectively}). \quad (1.9)$$

and when $m = \frac{n-2}{n+2}$ and $\rho_1 = 1$, then $\beta_0 = \frac{2}{\sqrt{n-2}}$, $\beta_1 = \frac{1}{2m}$, and (1.7) is equivalent to

$$\gamma^2 - \beta(n-2)\gamma + (n-2) = 0.$$

In [DKS] P. Daskalopoulos, J. King and N. Sesum, proved that when $m = \frac{n-2}{n+2}$, $n \geq 3$, $\rho_1 = 1$, and $\beta > \beta_0$, the radially symmetric solution v_λ of (1.3) in \mathbb{R}^n with $v_\lambda(0) = \lambda$ satisfies

$$v_\lambda(x) = \left(\frac{C_*}{|x|^2} \right)^{\frac{1}{1-m}} (1 - B|x|^{-\gamma} + o(|x|^{-\gamma})) \quad \text{as } |x| \rightarrow \infty \quad (1.10)$$

for some constants $B \in \mathbb{R}$,

$$C_* = \frac{2m(n-2-nm)}{(1-m)\rho_1}, \quad (1.11)$$

and $\gamma > 0$ where $\gamma = \gamma_2$ if $3 \leq n < 6$ and $\beta = \beta_1$, and $\gamma = \gamma_1$ otherwise. In this paper we will extend this second order asymptotic result to the case $0 < m < \frac{n-2}{n}$, $n \geq 3$ and $\rho_1 > 0$. For any $0 < m < \frac{n-2}{n}$, $n \geq 3$, and $\lambda > 0$, we will also extend Theorem 1.2 of [DKS] and prove the existence and uniqueness of radially symmetric singular solution $g_\lambda \in C^\infty(\mathbb{R}^n \setminus \{0\})$ of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies

$$\lim_{|x| \rightarrow 0} |x|^{\alpha/\beta} g_\lambda(x) = \lambda^{-\frac{\rho_1}{(1-m)\beta}}. \quad (1.12)$$

We also obtain higher order decay rate of g_λ as $|x| \rightarrow \infty$. Let

$$C(x) = \left(\frac{C_*}{r^2} \right)^{\frac{1}{1-m}}.$$

Then $C(x)$ is a solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$. In the papers [DS1], [BBDGV], etc. P. Daskalopoulos and N. Sesum, A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J.L. Vazquez, etc. obtain the asymptotic behaviour of the solution of (1.2) near the extinction time for $0 < m < \frac{n-2}{n}$, $n \geq 3$, when the initial value is sandwiched between two Barenblatt solutions. In this paper we will extend their results to initial values that satisfies other growth conditions.

More precisely we obtain the following main results in this paper.

Theorem 1.1. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda > 0$, $\beta \geq \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. Then there exists a radially symmetric solution g_λ of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies (1.12),*

$$g'_\lambda(r) \leq 0 \quad \forall r > 0 \quad (1.13)$$

and

$$\lambda^{-\frac{\rho_1}{(1-m)\beta}} \leq r^{\frac{\alpha}{\beta}} g_\lambda(r) \leq \lambda^{-\frac{\rho_1}{(1-m)\beta}} \exp\left(\frac{\beta C_0}{\rho_1} \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}}\right) \quad \forall r > 0. \quad (1.14)$$

for some constant $C_0 > 0$. Moreover if $\beta \geq \beta_1$, then the solution is unique.

Corollary 1.2. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 = 1$, $\lambda > 0$, $\beta \geq \frac{m}{n-2-nm}$ and $\alpha = \frac{2\beta+1}{1-m}$. Let g_λ be the radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies (1.12). Then for any $T > 0$ the function*

$$V_\lambda(x, t) = (T-t)^\alpha g_\lambda((T-t)^\beta x) \quad (1.15)$$

satisfies (1.1) in $\mathbb{R}^n \setminus \{0\}$ and

$$\lim_{|x| \rightarrow 0} |x|^{\frac{\alpha}{\beta}} V_\lambda(x, t) = \lambda^{-\frac{1}{(1-m)\beta}}.$$

Theorem 1.3. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, and $\lambda > 0$. Then there exists a constant $\beta_2 \geq \max(\beta_0, \beta_1)$ depending on n and m such that for any $\beta > \beta_2$, $\alpha = \frac{2\beta+\rho_1}{1-m}$, if v_λ is a radially symmetric solution of (1.3) in \mathbb{R}^n with $v_\lambda(0) = \lambda$, then (1.10) holds for some constants $B > 0$ and $\gamma = \gamma_1 > 0$ where γ_1 is given by (1.8).*

Theorem 1.4. Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda > 0$ and let $\beta_2 > 0$ be as in Theorem 1.3. Then for any $\beta > \beta_2$, $\alpha = \frac{2\beta+\rho_1}{1-m}$, if g_λ is the unique radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ which satisfies (1.12) that is given by Theorem 1.1, then

$$g_\lambda(x) = \left(\frac{C_*}{|x|^2} \right)^{\frac{1}{1-m}} (1 + B|x|^{-\gamma_1} + o(|x|^{-\gamma_1})) \quad \text{as } |x| \rightarrow \infty \quad (1.16)$$

holds for some constants $B > 0$ where γ_1 is given by (1.8).

By direct computation we also have the following inversion formula for the solution of (1.3).

Theorem 1.5. Let $n \geq 3$, $m = \frac{n-2}{n+2}$, $\alpha, \beta \in \mathbb{R}$. Let v be a radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ and $\widetilde{v}(\rho) = \rho^{-\frac{n-2}{m}} v(\rho^{-1})$. Then \widetilde{v} satisfies

$$\Delta \widetilde{v}^m + \alpha' \widetilde{v} + \beta' x \cdot \nabla \widetilde{v} = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

where $\alpha' = \alpha - \frac{n-2}{m}\beta$, $\beta' = -\beta$. If $\alpha = \frac{2\beta+\rho_1}{1-m}$ for some constant $\rho_1 > 0$, then $\alpha' = \frac{2\beta'+\rho_1}{1-m}$. Moreover $r^{\frac{\alpha}{\beta}} v(r) = \rho^{\frac{\alpha'}{\beta'}} \widetilde{v}(\rho)$ for all $r = \rho^{-1} > 0$.

For any solution u of (1.2) we let

$$\widetilde{u}(x, s) = (T - t)^{-\alpha} u((T - t)^{-\beta} x), \quad s = -\log(T - t). \quad (1.17)$$

Then \widetilde{u} satisfies

$$\widetilde{u}_s = \Delta \widetilde{u}^m + \alpha \widetilde{u} + \beta x \cdot \nabla \widetilde{u} \quad \text{in } \mathbb{R}^n \times (-\log T, \infty).$$

Theorem 1.6. Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $T > 0$, $\rho_1 = 1$, $\beta > \beta_1$ and $\alpha = \frac{2\beta+1}{1-m}$. Let ψ_λ be given by (1.5) and let u_0 satisfy $0 \leq u_0 \leq \psi_{\lambda_1}(x, 0)$ and

$$|u_0 - \psi_{\lambda_0}(x, 0)| \leq f(|x|) \in L^1(\mathbb{R}^n)$$

for some constants $\lambda_0 > 0$, $\lambda_1 > 0$, and radially symmetric function f . Let u be the solution of (1.2) and \widetilde{u} be given by (1.17). Then the rescaled solution $\widetilde{u}(\cdot, s)$ converges uniformly on every compact subset of \mathbb{R}^n to v_{λ_0} and in $L^1(\mathbb{R}^n)$ to v_{λ_0} as $s \rightarrow \infty$.

Theorem 1.7. Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $T > 0$, $\rho_1 = 1$, $\beta > \beta_1$ and $\alpha = \frac{2\beta+1}{1-m}$. Let V_λ be given by (1.15) and let u_0 satisfy

$$V_{\lambda_1}(x, 0) \leq u_0(x) \leq V_{\lambda_2}(x, 0) \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

and

$$u_0 - V_{\lambda_0}(x, 0) \in L^1(\mathbb{R}^n \setminus \{0\})$$

for some constants $\lambda_1 > \lambda_2 > 0$ and $\lambda_0 > 0$. Let u be a solution of (1.1) in $(\mathbb{R}^n \setminus \{0\}) \times (0, T)$ with initial value u_0 which satisfies

$$V_{\lambda_1}(x, t) \leq u(x, t) \leq V_{\lambda_2}(x, t) \quad \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, T)$$

and let \widetilde{u} be given by (1.17). Then the rescaled solution $\widetilde{u}(\cdot, s)$ converges uniformly on every compact subset of $\mathbb{R}^n \setminus \{0\}$ and in $L^1(\mathbb{R}^n \setminus \{0\})$ to g_{λ_0} as $s \rightarrow \infty$. Moreover,

$$\|\widetilde{u}(\cdot, s) - g_{\lambda_0}\|_{L^1(\mathbb{R}^n \setminus \{0\})} \leq e^{-(n\beta-\alpha)s} \|u_0 - V_{\lambda_0}(\cdot, 0)\|_{L^1(\mathbb{R}^n \setminus \{0\})} \quad \forall s > -\log T.$$

The plan of the paper is as follows. In section two we will prove Theorem 1.1. We will prove Theorem 1.3 and Theorem 1.4 in section three and four respectively. Finally we will sketch the proof of Theorem 1.6 and Theorem 1.7 in section five.

Unless stated otherwise we will assume that $n \geq 3, 0 < m < \frac{n-2}{n}, \rho_1 > 0, \lambda > 0, \beta > \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$, for the rest of the paper. For any $R > 0$, we let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. For any $T > 0$ and domain $\Omega \subset \mathbb{R}^n$, we say that u is a solution of (1.1) in $\Omega \times (0, T)$ if u is a smooth positive solution of (1.1) in $\Omega \times (0, T)$. For any $0 \leq u_0 \in L^1_{loc}(\Omega)$ we say that u is a solution of (1.1) in $\Omega \times (0, T)$ with initial value u_0 if u is a solution of (1.1) in $\Omega \times (0, T)$ with $u(\cdot, t) \rightarrow u_0$ in $L^1_{loc}(\Omega)$ as $t \rightarrow 0$.

2 Existence of blow-up solutions

In this section we will prove Theorem 1.1. We first start with a technical lemma.

Lemma 2.1. *Let $\beta \geq \beta_1$ and g_λ be a radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ which satisfies (1.12). Then g_λ satisfies*

$$r^{n-1}(v^m)'(r) + \beta r^n v(r) = (n\beta - \alpha) \int_0^r v(\rho) \rho^{n-1} d\rho \quad \forall r > 0 \quad (2.1)$$

if $\beta > \beta_1$ and g_λ satisfies

$$r^{n-1}(v^m)'(r) + \beta r^n v(r) = \beta \lambda^{-\frac{\rho_1}{(1-m)\beta}} \quad \forall r > 0 \quad \text{if } \beta = \beta_1. \quad (2.2)$$

Proof: We first claim that there exists a sequence of positive numbers $\{\xi_i\}_{i=1}^\infty$, $\xi_i \rightarrow 0$ as $i \rightarrow \infty$, such that

$$\lim_{i \rightarrow \infty} \xi_i^{\frac{\alpha}{\beta}+1} g'_\lambda(\xi_i) = -\frac{\alpha}{\beta} \lambda^{-\frac{\rho_1}{(1-m)\beta}}. \quad (2.3)$$

In order to proof the claim we choose a sequence of positive numbers $\{r_i\}_{i=1}^\infty$ such that $r_i \rightarrow 0$ as $i \rightarrow \infty$. Then by (1.12) and the mean value theorem for any $i \in \mathbb{Z}^+$ there exists $\xi_i \in (r_i/2, r_i)$ such that

$$\begin{aligned} \frac{r_i^{\frac{\alpha}{\beta}+1} g_\lambda(r_i) - (r_i/2)^{\frac{\alpha}{\beta}+1} g_\lambda(r_i/2)}{r_i/2} &= \xi_i^{\frac{\alpha}{\beta}+1} g'_\lambda(\xi_i) + \left(1 + \frac{\alpha}{\beta}\right) \xi_i^{\frac{\alpha}{\beta}} g_\lambda(\xi_i) \\ \Rightarrow \left| \xi_i^{\frac{\alpha}{\beta}+1} g'_\lambda(\xi_i) \right| &\leq 2r_i^{\frac{\alpha}{\beta}} g_\lambda(r_i) + (r_i/2)^{\frac{\alpha}{\beta}} g_\lambda(r_i/2) + \left(1 + \frac{\alpha}{\beta}\right) \xi_i^{\frac{\alpha}{\beta}} g_\lambda(\xi_i) \leq C \quad \forall i \in \mathbb{Z}^+ \end{aligned}$$

for some constant $C > 0$. Hence the sequence $\{\xi_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $\xi_i^{\frac{\alpha}{\beta}+1} g'_\lambda(\xi_i)$ converges to some constant as $i \rightarrow \infty$. Then by (1.12) and L'Hospital's Rule,

$$\lambda^{-\frac{\rho_1}{(1-m)\beta}} = \lim_{i \rightarrow \infty} \xi_i^{\frac{\alpha}{\beta}} g_\lambda(\xi_i) = \lim_{i \rightarrow \infty} \frac{g_\lambda(\xi_i)}{\xi_i^{-\frac{\alpha}{\beta}}} = \lim_{i \rightarrow \infty} \frac{g'_\lambda(\xi_i)}{-\frac{\alpha}{\beta} \xi_i^{-\frac{\alpha}{\beta}-1}} = -\frac{\beta}{\alpha} \lim_{i \rightarrow \infty} \xi_i^{\frac{\alpha}{\beta}+1} g'_\lambda(\xi_i)$$

and (2.3) follows. By (1.3) g_λ satisfies

$$\begin{aligned} (v^m)'' + \frac{n-1}{r}(v^m)' + \alpha v + \beta r v' &= 0, \quad v > 0 \quad \forall r > 0 \\ \Leftrightarrow (r^{n-1}(v^m)')' + \alpha r^{n-1}v + \beta r^n v' &= 0, \quad v > 0 \quad \forall r > 0. \end{aligned} \quad (2.4)$$

Integrating (2.4) over (ξ, r) ,

$$\begin{aligned} & r^{n-1}(g_\lambda^m)'(r) + \beta r^n g_\lambda(r) \\ &= \xi^{n-1}(g_\lambda^m)'(\xi) + \beta \xi^n g_\lambda(\xi) + (n\beta - \alpha) \int_\xi^r g_\lambda(\rho) \rho^{n-1} d\rho \quad \forall r > \xi > 0 \\ &= m \xi^{n-2-\frac{m\alpha}{\beta}} (\xi^{\frac{\alpha}{\beta}} g_\lambda(\xi))^{m-1} (\xi^{\frac{\alpha}{\beta}+1} g_\lambda'(\xi)) + \beta \xi^{\frac{n\beta-\alpha}{\beta}} (\xi^{\frac{\alpha}{\beta}} g_\lambda(\xi)) + (n\beta - \alpha) \int_\xi^r g_\lambda(\rho) \rho^{n-1} d\rho \quad \forall r > \xi > 0 \end{aligned} \quad (2.5)$$

Since $\beta \geq \beta_1$, $n - 2 - \frac{m\alpha}{\beta} > 0$. Putting $\xi = \xi_i$ in (2.5) and letting $i \rightarrow \infty$, by (1.9), (1.12), and (2.3), we get that g_λ satisfies (2.1) if $\beta > \beta_1$ and g_λ satisfies (2.2) if $\beta = \beta_1$. \square

Similarly we have the following lemma.

Lemma 2.2. *Let $\beta > 0$. If v_λ is a radially symmetric solution of (1.3) in \mathbb{R}^n which satisfies $v_\lambda(0) = \lambda$, then v_λ satisfies (2.1).*

Lemma 2.3. *Let $\beta \geq \beta_1$ and $\lambda_2 > \lambda_1 > 0$. Then*

$$g_{\lambda_2}(r) < g_{\lambda_1}(r) \quad \forall r > 0.$$

Proof: We will use a modification of the proof of Lemma 2.3 of [HuK] to proof the lemma. Since $\lambda_2 > \lambda_1$, by (1.12) there exists a constant $r_1 > 0$ such that $g_{\lambda_2}(r) < g_{\lambda_1}(r)$ for any $0 < r \leq r_1$. Let $(0, r_0)$ be the maximal interval such that

$$g_{\lambda_2}(r) < g_{\lambda_1}(r) \quad \forall 0 < r < r_0. \quad (2.6)$$

Suppose $r_0 < \infty$. Then $g_{\lambda_2}(r_0) = g_{\lambda_1}(r_0)$ and $g_{\lambda_2}'(r_0) \geq g_{\lambda_1}'(r_0)$. If $\beta > \beta_1$, then by Lemma 2.1 both g_{λ_1} and g_{λ_2} satisfy (2.1). Hence if $\beta > \beta_1$, by (1.9), (2.6), and Lemma 2.1,

$$\begin{aligned} r_0^{n-1}(g_{\lambda_2}^m)'(r_0) &= -\beta r_0^n g_{\lambda_2}(r_0) + (n\beta - \alpha) \int_0^{r_0} g_{\lambda_2}(\rho) \rho^{n-1} d\rho \\ &< -\beta r_0^n g_{\lambda_1}(r_0) + (n\beta - \alpha) \int_0^{r_0} g_{\lambda_1}(\rho) \rho^{n-1} d\rho \\ &= r_0^{n-1}(g_{\lambda_1}^m)'(r_0). \end{aligned}$$

If $\beta = \beta_1$, by Lemma 2.1 both g_{λ_1} and g_{λ_2} satisfy (2.2). Hence by (2.6),

$$r_0^{n-1}(g_{\lambda_2}^m)'(r_0) = -r_0^n g_{\lambda_2}(r_0) + \beta \lambda_2^{-\frac{\rho_1}{(1-m)\beta}} < -r_0^n g_{\lambda_1}(r_0) + \beta \lambda_1^{-\frac{\rho_1}{(1-m)\beta}} = r_0^{n-1}(g_{\lambda_1}^m)'(r_0).$$

Hence for any $\beta \geq \beta_1$,

$$\begin{aligned} & (g_{\lambda_2}^m)'(r_0) < (g_{\lambda_1}^m)'(r_0) \\ \Rightarrow & m g_{\lambda_2}^{m-1}(r_0) g'_{\lambda_2}(r_0) < m g_{\lambda_1}^{m-1}(r_0) g'_{\lambda_1}(r_0) \\ \Rightarrow & g'_{\lambda_2}(r_0) < g'_{\lambda_1}(r_0) \end{aligned}$$

and contradiction arises. Thus $r_0 = \infty$ and the lemma follows. \square

By direct computation $C(r)$ satisfies (2.1) and (2.4). Since $C(r) \rightarrow \infty$ as $r \rightarrow 0$ and $r^{\frac{\alpha}{\beta}} C(r) \rightarrow 0$ as $r \rightarrow 0$, by Lemma 2.1, Lemma 2.2, and an argument similar to the proof of Lemma 2.3 we have the following result.

Lemma 2.4. *Let $\beta \geq \beta_1$. Then*

$$v_{\lambda_1}(r) < v_{\lambda_2}(r) \quad \forall r \geq 0, \lambda_2 > \lambda_1 > 0$$

and

$$v_{\lambda}(r) < C(r) < g_{\lambda}(r) \quad \forall r > 0, \lambda > 0. \quad (2.7)$$

Theorem 2.5. *Let $\beta \geq \beta_1$ and $\lambda > 0$. Suppose g_1 and g_2 are two radially symmetric solutions of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies (1.12). Then $g_1 = g_2$ on $\mathbb{R}^n \setminus \{0\}$.*

Proof: We choose a monotone decreasing sequence $\lambda_i > 1$ for all $i \in \mathbb{Z}^+$ such that $\lambda_i \rightarrow 1$ as $i \rightarrow \infty$. Let

$$w_i(r) = \lambda_i^{\frac{2}{1-m}} g_1(\lambda_i r)$$

Then w_i satisfies (1.3) in $\mathbb{R}^n \setminus \{0\}$ and

$$\lim_{r \rightarrow 0} r^{\frac{\alpha}{\beta}} w_i(r) = \lambda_i^{-\frac{\rho_1}{(1-m)\beta}} \lim_{r \rightarrow 0} (\lambda_i r)^{\frac{\alpha}{\beta}} g_i(\lambda_i r) = (\lambda_i \lambda)^{-\frac{\rho_1}{(1-m)\beta}} < (\lambda_{i+1} \lambda)^{-\frac{\rho_1}{(1-m)\beta}} < \lambda^{-\frac{\rho_1}{(1-m)\beta}} \quad \forall i \in \mathbb{Z}^+. \quad (2.8)$$

By (2.8) and Lemma 2.3,

$$\begin{aligned} & w_i(r) < w_{i+1}(r) < g_2(r) \quad \forall r > 0, i \in \mathbb{Z}^+ \\ \Rightarrow & g_1(r) \leq g_2(r) \quad \forall r > 0 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Similarly by interchanging the role of g_1 and g_2 in the above argument we get $g_1(r) \geq g_2(r)$ for all $r > 0$. Hence $g_1 = g_2$ on $\mathbb{R}^n \setminus \{0\}$ and the theorem follows. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: By Theorem 2.5 we only need to prove existence of radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ or solution of (2.4) that satisfies (1.12), (1.13) and (1.14), when $\beta \geq \frac{m\rho_1}{n-2-nm}$. Let $\beta \geq \frac{m\rho_1}{n-2-nm}$.

Claim 1: For any $\xi_0 > 0$, there exists a radially symmetric solution g of (1.3) in $\mathbb{R}^n \setminus B_{\xi_0}$ which satisfies

$$g(\xi_0) = \lambda^{-\frac{\rho_1}{(1-m)\beta}} \xi_0^{-\frac{\alpha}{\beta}} \quad \text{and} \quad g'(\xi_0) = -\frac{\alpha}{\beta} \lambda^{-\frac{\rho_1}{(1-m)\beta}} \xi_0^{-\frac{\alpha}{\beta}-1}. \quad (2.9)$$

In order to prove this claim we first observe that by the standard O.D.E. theory there exist $\varepsilon > 0$ and a solution g of (2.4) in $(\xi_0, \xi_0 + \varepsilon)$ which satisfies (2.9). Let (ξ_0, R_0) be the maximal interval of existence of solution of (2.4) which satisfies (2.9). Let $\bar{w}(r) = r^{\frac{\alpha}{\beta}} g(r)$, $h_1(r) = g(r) + (\beta/\alpha)rg'(r)$, and

$$f_1(r) = g(r)^{m-1} \exp \left(\frac{\beta}{m} \int_{\xi_0}^r \rho g(\rho)^{1-m} d\rho \right).$$

By (2.9), $h_1(\xi_0) = 0$. As observed in [Hs1], h_1 satisfies

$$h_1' + \left(\frac{n-2-\frac{m\alpha}{\beta}}{r} - (1-m)\frac{g'}{g} + \frac{\beta}{m}rg^{1-m} \right) h_1 = \frac{n-2-\frac{m\alpha}{\beta}}{r} g \geq 0$$

in (ξ_0, R_0) . Hence

$$\begin{aligned} & (r^{n-2-\frac{m\alpha}{\beta}} f_1(r) h_1(r))' \geq 0 & \forall \xi_0 < r < R_0 \\ \Rightarrow & r^{n-2-\frac{m\alpha}{\beta}} f_1(r) h_1(r) \geq \xi_0^{n-2-\frac{m\alpha}{\beta}} f_1(\xi_0) h_1(\xi_0) = 0 & \forall \xi_0 < r < R_0 \\ \Rightarrow & h_1(r) \geq 0 & \forall \xi_0 < r < R_0 \\ \Rightarrow & \bar{w}'(r) = \frac{\alpha}{\beta} r^{\frac{\alpha}{\beta}-1} h_1(r) \geq 0 & \forall \xi_0 < r < R_0 \end{aligned} \quad (2.10)$$

$$\Rightarrow \bar{w}(r) = r^{\frac{\alpha}{\beta}} g(r) \geq \bar{w}(\xi_0) = \lambda^{-\frac{\rho_1}{(1-m)\beta}} \quad \forall \xi_0 < r < R_0. \quad (2.11)$$

By (2.4), (2.9), and (2.10),

$$\begin{aligned} & (r^{n-1}(g^m)')' = -\alpha r^{n-1} h_1(r) \leq 0 & \forall \xi_0 < r < R_0 \\ \Rightarrow & r^{n-1}(g^m)'(r) \leq \xi_0^{n-1}(g^m)'(\xi_0) < 0 & \forall \xi_0 < r < R_0 \\ \Rightarrow & g'(r) < 0 & \forall \xi_0 \leq r < R_0. \end{aligned} \quad (2.12)$$

By (2.11) and (2.12),

$$\lambda^{-\frac{\rho_1}{(1-m)\beta}} r^{-\frac{\alpha}{\beta}} < g(r) < g(\xi_0) \quad \forall \xi_0 < r < R_0. \quad (2.13)$$

Suppose $R_0 < \infty$. Since g satisfies (2.5) with $\xi = \xi_0$, $r \in (\xi_0, R_0)$, by (2.13), there exists a constant $C > 0$ independent of R_0 such that

$$|g'(r)| \leq C(1 + R_0^n) \quad \forall \xi_0 < r < R_0. \quad (2.14)$$

By (2.13) and (2.14) we can extend g to a solution of (2.4) in (ξ_0, R_1) that satisfies (2.9) for some $R_1 > R_0$. This contradicts the choice of R_0 . Hence $R_0 = \infty$ and claim 1 follows.

By claim 1 for any $i \in \mathbb{Z}^+$ there exists a radially symmetric solution g_i of (1.3) in $\mathbb{R}^n \setminus B_{1/i}$ or equivalently a solution of (2.4) in $(1/i, \infty)$ which satisfies

$$g_i(1/i) = \lambda^{-\frac{\rho_1}{(1-m)\beta}} i^{\frac{\alpha}{\beta}} \quad \text{and} \quad g_i'(1/i) = -\frac{\alpha}{\beta} \lambda^{-\frac{\rho_1}{(1-m)\beta}} i^{\frac{\alpha}{\beta}+1}. \quad (2.15)$$

Let $w_i(r) = r^{\frac{\alpha}{\beta}} g_i(r)$. By the proof of claim 1,

$$g'_i(r) < 0 \quad \forall r > 1/i, i \in \mathbb{Z}^+ \quad (2.16)$$

and

$$\begin{aligned} w'_i(1/i) &= 0 \quad \text{and} \quad w'_i(r) \geq 0 \quad \forall r > 1/i, i \in \mathbb{Z}^+ \\ \Rightarrow w_i(r) &\geq w_i(1/i) = \lambda^{-\frac{\rho_1}{(1-m)\beta}} \quad \forall r > 1/i, i \in \mathbb{Z}^+ \end{aligned} \quad (2.17)$$

$$\Rightarrow g_i(r) \geq \lambda^{-\frac{\rho_1}{(1-m)\beta}} r^{-\frac{\alpha}{\beta}} \quad \forall r > 1/i, i \in \mathbb{Z}^+. \quad (2.18)$$

By direct computation (cf. [Hs1]),

$$\left(\frac{w'_i}{w_i}\right)' + \frac{n-1-\frac{2m\alpha}{\beta}}{r} \cdot \frac{w'_i}{w_i} + m \left(\frac{w'_i}{w_i}\right)^2 + \frac{\beta r^{-1-\frac{\rho_1}{\beta}} w'_i}{m w_i^m} = \frac{\alpha}{\beta} \cdot \frac{n-2-\frac{m\alpha}{\beta}}{r^2} \quad \forall r > 1/i, i \in \mathbb{Z}^+. \quad (2.19)$$

Let $s = \log r$ and $z_i = w_{i,s}/w_i$. Then $z_i(-\log i) = 0$ and $z_i(s) \geq 0$ for any $s > -\log i$. By (2.19) and a direct computation,

$$z_{i,s} + \left(n-2-\frac{2m\alpha}{\beta}\right) z_i + m z_i^2 + \frac{\beta}{m} e^{-\frac{\rho_1}{\beta}s} w_i^{1-m} z_i = \frac{\alpha}{\beta} \left(n-2-\frac{m\alpha}{\beta}\right) \quad \forall s > -\log i, i \in \mathbb{Z}^+ \quad (2.20)$$

By (2.17) and (2.20),

$$\begin{aligned} z_{i,s} + m(z_i^2 + 2C_1 z_i) + \frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s} z_i &\leq C_2 \quad \forall s > -\log i, i \in \mathbb{Z}^+ \\ \Rightarrow z_{i,s} + \frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s} z_i &\leq z_{i,s} + m(z_i + C_1)^2 + \frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s} z_i \leq C'_2 \quad \forall s > -\log i, i \in \mathbb{Z}^+ \end{aligned} \quad (2.21)$$

where $C_1 = \frac{1}{2m} \left(n-2-\frac{2m\alpha}{\beta}\right)$, $C_2 = \frac{\alpha}{\beta} \left(n-2-\frac{m\alpha}{\beta}\right) > 0$ and $C'_2 = C_2 + mC_1^2$. Let

$$a(s) = -\frac{\beta^2}{m\rho_1} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s}.$$

Then by (2.21),

$$z_i(s) \leq C'_2 \int_{-\log i}^s e^{a(s')-a(s)} ds' \quad \forall s > -\log i, i \in \mathbb{Z}^+. \quad (2.22)$$

By the mean value theorem for any $s > s' > -\log i$ there exists a constant $s_1 \in (s', s)$ such that

$$a(s') - a(s) = \frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s_1} (s' - s) \leq \frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s} (s' - s). \quad (2.23)$$

By (2.22) and (2.23),

$$\begin{aligned}
z_i(s) &\leq C'_2 \int_{-\log i}^s \exp\left(\frac{\beta}{m} \lambda^{-\frac{\rho_1}{\beta}} e^{-\frac{\rho_1}{\beta}s} (s' - s)\right) ds' \leq C_0 \lambda^{\frac{\rho_1}{\beta}} e^{\frac{\rho_1}{\beta}s} \quad \forall s > -\log i, i \in \mathbb{Z}^+ \\
\Rightarrow \quad \frac{r w_{i,r}}{w_i} &\leq C_0 \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}} \quad \forall r > 1/i, i \in \mathbb{Z}^+ \\
\Rightarrow \quad w_i(r) &\leq \lambda^{-\frac{\rho_1}{(1-m)\beta}} \exp\left(\frac{\beta C_0}{\rho_1} \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}}\right) \quad \forall r > 1/i, i \in \mathbb{Z}^+ \\
\Rightarrow \quad g_i(r) &\leq \lambda^{-\frac{\rho_1}{(1-m)\beta}} r^{-\frac{\alpha}{\beta}} \exp\left(\frac{\beta C_0}{\rho_1} \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}}\right) \quad \forall r > 1/i, i \in \mathbb{Z}^+ \tag{2.24}
\end{aligned}$$

where $C_0 = \frac{mC'_2}{\beta}$. By (2.18) and (2.24), the equation (1.3) for the sequence $\{g_i\}_{i=1}^\infty$ is uniformly elliptic on every compact subset of $\mathbb{R}^n \setminus \{0\}$. Hence by standard Schauder's estimates [GT] the sequence $\{g_i\}_{i=1}^\infty$ is uniformly continuous in $C^2(K)$ for any compact set $K \subset \mathbb{R}^n \setminus \{0\}$. By the Ascoli Theorem and a diagonalization argument the sequence $\{g_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^2(K)$ for any compact set $K \subset \mathbb{R}^n \setminus \{0\}$ to some function $g_\lambda \in C^2(\mathbb{R}^n \setminus \{0\})$ as $i \rightarrow \infty$. Then g_λ is a radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$. Letting $i \rightarrow \infty$ in (2.18) and (2.24),

$$\begin{aligned}
\lambda^{-\frac{\rho_1}{(1-m)\beta}} r^{-\frac{\alpha}{\beta}} &\leq g_\lambda(r) \leq \lambda^{-\frac{\rho_1}{(1-m)\beta}} r^{-\frac{\alpha}{\beta}} \exp\left(\frac{\beta C_0}{\rho_1} \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}}\right) \quad \forall r > 0, i \in \mathbb{Z}^+ \\
\Rightarrow \quad \lambda^{-\frac{\rho_1}{(1-m)\beta}} &\leq r^{\frac{\alpha}{\beta}} g_\lambda(r) \leq \lambda^{-\frac{\rho_1}{(1-m)\beta}} \exp\left(\frac{\beta C_0}{\rho_1} \lambda^{\frac{\rho_1}{\beta}} r^{\frac{\rho_1}{\beta}}\right) \quad \forall r > 0, i \in \mathbb{Z}^+. \tag{2.25}
\end{aligned}$$

Letting $r \rightarrow 0$ in (2.16) and (2.25) we get (1.12) and (1.13) and the theorem follows. \square

Corollary 2.6. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\beta \geq \beta_1$ and $\alpha = \frac{2\beta + \rho_1}{1-m}$. Let g_{λ_1} , g_{λ_2} , be two radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies (1.12) with λ being replaced by λ_1 , λ_2 respectively. Then*

$$g_{\lambda_2}(x) = (\lambda_2/\lambda_1)^{\frac{2}{1-m}} g_{\lambda_1}((\lambda_2/\lambda_1)x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Proof. Let

$$\widetilde{g}(x) = (\lambda_2/\lambda_1)^{\frac{2}{1-m}} g_{\lambda_1}((\lambda_2/\lambda_1)x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Then \widetilde{g} is a solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ and

$$\lim_{|x| \rightarrow 0} |x|^{\frac{\alpha}{\beta}} \widetilde{g}(x) = (\lambda_2/\lambda_1)^{-\frac{\rho_1}{(1-m)\beta}} \lim_{|x| \rightarrow 0} ((\lambda_2/\lambda_1)|x|)^{\frac{\alpha}{\beta}} g_{\lambda_1}((\lambda_2/\lambda_1)x) = \lambda_2^{-\frac{\rho_1}{(1-m)\beta}}.$$

Hence by Theorem 2.5, $\widetilde{g}(x) \equiv g_{\lambda_2}(x)$ on $\mathbb{R}^n \setminus \{0\}$ and the corollary follows. \square

By a similar argument we have the following corollary.

Corollary 2.7. Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\beta \geq \beta_1$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. Let $v_{\lambda_1}, v_{\lambda_2}$, be two radially symmetric solution of (1.3) in \mathbb{R}^n with $v_{\lambda_1}(0) = \lambda_1, v_{\lambda_2}(0) = \lambda_2$. Then

$$v_{\lambda_2}(x) = \frac{\lambda_2}{\lambda_1} v_{\lambda_1}((\lambda_2/\lambda_1)^{\frac{1-m}{2}} x) \quad \forall x \in \mathbb{R}^n. \quad (2.26)$$

Note that by an argument similar to the proof of [Hs3],

$$\lim_{|x| \rightarrow \infty} |x|^2 g_\lambda(x)^{1-m} = \frac{2m(n-2-nm)}{(1-m)\rho_1}. \quad (2.27)$$

Then by (2.27), Lemma 2.3, Corollary 2.6 and an argument similar to the proof of Corollary 1.3 of [Hs3] but with g_λ replacing v_λ in the proof there we get the following corollary.

Corollary 2.8. Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\lambda > 0$, $\beta \geq \beta_1$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. Let g_λ be the radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ that satisfies (1.12). Then $g_\lambda(x)$ decreases and converges uniformly on $\mathbb{R}^n \setminus B_R$ to $C(x)$ for any $R > 0$ as $\lambda \rightarrow \infty$.

3 Second order asymptotic of self-similar solutions

In this section we will use a modification of the proof of [DKS] to prove Theorem 1.3. Let $s = \log r$ and

$$\bar{q}(s) = \left(r^{\frac{2}{1-m}} v_\lambda(r) \right)^m.$$

Then by the computation in section 3 of [Hs1],

$$\bar{q}_{ss} + \frac{n-2-(n+2)m}{1-m} \bar{q}_s + \beta(\bar{q}^{\frac{1}{m}})_s + \frac{\rho_1}{1-m} \bar{q}^{\frac{1}{m}} - \frac{2m(n-2-nm)}{(1-m)^2} \bar{q} = 0 \quad \text{in } \mathbb{R}. \quad (3.1)$$

Let $q(s) = \bar{q}(s)/C_*^{\frac{m}{1-m}}$ where C_* is given by (1.11). Then by (3.1),

$$q_{ss} + \left(\frac{n-2-(n+2)m}{1-m} + \frac{\beta C_*}{m} q^{\frac{1}{m}-1} \right) q_s + \frac{2m(n-2-nm)}{(1-m)^2} (q^{\frac{1}{m}} - q) = 0 \quad \text{in } \mathbb{R}. \quad (3.2)$$

We now linearize (3.2) around the constant 1 solution by setting $q = 1 + w$ in (3.2). Then w satisfies

$$w_{ss} + \left(\frac{n-2-(n+2)m}{1-m} + \frac{\beta C_*}{m} (1+w)^{\frac{1}{m}-1} \right) w_s + \frac{2m(n-2-nm)}{(1-m)^2} ((1+w)^{\frac{1}{m}} - 1 - w) = 0 \quad \text{in } \mathbb{R} \quad (3.3)$$

and $w(s) > -1$ for all $s \in \mathbb{R}$. Then the linearized operator of (3.3) around $w = 0$ is

$$Lw := w_{ss} + \left(\frac{n-2-(n+2)m}{1-m} + \frac{2\beta(n-2-nm)}{(1-m)\rho_1} \right) w_s + \frac{2(n-2-nm)}{(1-m)} w.$$

Note that the function $e^{-\gamma s}$ is a solution of $Lw = 0$ if and only if γ satisfies (1.7) whose two roots $\gamma_2 > \gamma_1 > 0$ if $\beta > \beta_0$. We now rewrite (3.3) as

$$w_{ss} + \left(\frac{n-2-(n+2)m}{1-m} + \frac{2\beta(n-2-nm)}{(1-m)\rho_1} \right) w_s + \frac{2(n-2-nm)}{(1-m)} w = f \quad (3.4)$$

where

$$f(s) = -\frac{2m(n-2-nm)}{1-m} \left\{ \frac{\beta}{m\rho_1} \left((1+w)^{\frac{1}{m}-1} - 1 \right) w_s + \frac{1}{1-m} \left((1+w)^{\frac{1}{m}} - 1 - \frac{1}{m} w \right) \right\}.$$

Let

$$\phi(z) = (1+z)^{\frac{1}{m}} - 1 - \frac{1}{m}z \quad \forall z > -1$$

and

$$\widetilde{\phi}(s) = \phi(w(s)).$$

Then $\phi(z)$ is a non-negative convex function satisfying

$$a_1 z^2 \leq \phi(z) \leq a_2 z^2 \quad \forall |z| \leq 1/10 \quad (3.5)$$

for some constants $a_2 > a_1 > 0$ and

$$f(s) = -\frac{2m(n-2-nm)}{1-m} \left\{ \frac{\beta}{\rho_1} \widetilde{\phi}'(s) + \frac{1}{1-m} \widetilde{\phi}(s) \right\}.$$

Since $q(s) \rightarrow 0$ as $s \rightarrow -\infty$, $w(s) \rightarrow -1$ as $s \rightarrow -\infty$. By the result of [Hs3], $q(s) \rightarrow 1$ as $s \rightarrow \infty$. Hence $w(s) \rightarrow 0$ as $s \rightarrow \infty$. Let $s_i = -i^2$ for all $i \in \mathbb{Z}^+$. Then by the mean value theorem for any $i \in \mathbb{Z}^+$ there exists a constant $s'_i \in (s_{i+1}, s_i)$ such that

$$\begin{aligned} |w_s(s'_i)| &= \left| \frac{w(s_i) - w(s_{i+1})}{s_i - s_{i+1}} \right| \leq \frac{|w(s_i)| + |w(s_{i+1})|}{2i+1} \\ \Rightarrow \lim_{i \rightarrow \infty} |w_s(s'_i)| &= 0. \end{aligned} \quad (3.6)$$

Since γ_1, γ_2 are roots of (1.7),

$$\gamma_1 + \gamma_2 = \left(\frac{n-2-(n+2)m}{1-m} + \frac{2\beta(n-2-nm)}{(1-m)\rho_1} \right). \quad (3.7)$$

Multiplying (3.4) by $e^{\gamma_1 s}$ and integrating over (s'_i, s) , by (3.7) and integration by parts,

$$\int_{s'_i}^s e^{\gamma_1 t} f(t) dt = e^{\gamma_1 s} w_s(s) - e^{\gamma_1 s'_i} w_s(s'_i) + \gamma_2 (e^{\gamma_1 s} w(s) - e^{\gamma_1 s'_i} w(s'_i)). \quad (3.8)$$

Letting $i \rightarrow \infty$ in (3.8), by (3.6),

$$w_s(s) + \gamma_2 w(s) = e^{-\gamma_1 s} \int_{-\infty}^s e^{\gamma_1 t} f(t) dt. \quad (3.9)$$

Similarly,

$$w_s(s) + \gamma_1 w(s) = e^{-\gamma_2 s} \int_{-\infty}^s e^{\gamma_2 t} f(t) dt. \quad (3.10)$$

Subtracting (3.10) from (3.9),

$$w(s) = \frac{1}{\gamma_2 - \gamma_1} \left\{ e^{-\gamma_1 s} \int_{-\infty}^s e^{\gamma_1 t} f(t) dt - e^{-\gamma_2 s} \int_{-\infty}^s e^{\gamma_2 t} f(t) dt \right\} \quad \forall s \in \mathbb{R}. \quad (3.11)$$

We are now ready to proof Theorem 1.3.

Proof of Theorem 1.3: By (3.11) and integration by parts,

$$w(s) = -M_0 \left\{ A_1(\beta) e^{-\gamma_1 s} \int_{-\infty}^s e^{\gamma_1 t} \tilde{\phi}(t) dt - A_2(\beta) e^{-\gamma_2 s} \int_{-\infty}^s e^{\gamma_2 t} \tilde{\phi}(t) dt \right\} \quad \forall s \in \mathbb{R} \quad (3.12)$$

where

$$M_0 = \frac{2m(n-2-nm)}{(1-m)(\gamma_2 - \gamma_1)} \quad (3.13)$$

and

$$A_i(\beta) = \frac{1}{1-m} - \frac{\beta \gamma_i}{\rho_1}, \quad i = 1, 2. \quad (3.14)$$

Let $c_2 = \left(1 - \frac{m}{2}\right)^2$ and

$$b_0 = \max \left(2 \sqrt{\frac{2(1-m)}{(n-2-nm)(1-c_2^2)}}, \frac{\sqrt{2}}{\sqrt{n-2-nm}} \right).$$

If $0 < m \leq \frac{n-2}{n+2}$, we choose $a_0 = b_0$. If $\frac{n-2}{n+2} < m < \frac{n-2}{n}$, we will choose $a_0 > 0$ later such that it is strictly greater than b_0 . Let

$$\beta_2 = \max(a_0 \rho_1, \beta_0, \beta_1)$$

and $\beta > \beta_2$. Then

$$\sqrt{A(\beta)^2 - 8(n-2-nm)(1-m)} \geq c_2 A(\beta).$$

Hence

$$\begin{aligned} A_1(\beta) &= \frac{1}{1-m} \left\{ 1 + \frac{\beta}{2\rho_1} \left(\sqrt{A(\beta)^2 - 8(n-2-nm)(1-m)} - A(\beta) \right) \right\} \\ &= \frac{1}{1-m} \left\{ 1 - \frac{4\beta(n-2-nm)(1-m)}{\rho_1 \left(\sqrt{A(\beta)^2 - 8(n-2-nm)(1-m)} + A(\beta) \right)} \right\} \\ &\geq \frac{1}{1-m} \left\{ 1 - \frac{4\beta(n-2-nm)(1-m)}{\rho_1(1+c_2)A(\beta)} \right\} \end{aligned} \quad (3.15)$$

$$\begin{aligned} &\geq \frac{1}{1-m} \left(1 - \frac{2(1-m)}{1 + (1 - (m/2))^2} \right) && \text{if } 0 < m \leq \frac{n-2}{n+2} \\ &> 0. && \text{if } 0 < m \leq \frac{n-2}{n+2} \end{aligned} \quad (3.16)$$

Letting $\beta \rightarrow \infty$ in (3.15),

$$\liminf_{\beta \rightarrow \infty} A_1(\beta) \geq \frac{1}{1-m} \left\{ 1 - \frac{2(1-m)}{1+c_2} \right\} > 0.$$

Hence for $\frac{n-2}{n+2} < m < \frac{n-2}{n}$, $n \geq 3$, we can choose $a_0 > b_0$ such that

$$A_1(\beta) > 0 \quad \forall \beta > \beta_2. \quad (3.17)$$

Similarly,

$$\begin{aligned} A_2(\beta) &= \frac{1}{1-m} \left\{ 1 - \frac{\beta}{2\rho_1} \left(\sqrt{A(\beta)^2 - 8(n-2-nm)(1-m)} + A(\beta) \right) \right\} \\ &\leq \frac{1}{1-m} \left\{ 1 - \frac{\beta A(\beta)}{2\rho_1} \right\} \\ &\leq \frac{1}{1-m} \left\{ 1 - \frac{\beta^2}{2\rho_1^2} (n-2-nm) \right\} \\ &< 0 \quad \forall \beta > \beta_2. \end{aligned} \quad (3.18)$$

Since $e^{-\gamma_1(s-t)} \geq e^{-\gamma_2(s-t)}$ for all $t \in (-\infty, s)$, by (3.12), (3.16), (3.17) and (3.18),

$$0 > w(s) > -M_0(A_1(\beta) + |A_2(\beta)|)e^{-\gamma_1 s} \int_{-\infty}^s e^{\gamma_1 t} \widetilde{\phi}(t) dt. \quad (3.19)$$

By (3.19) and an argument similar to the proof of Lemma 3.2 of [DKS],

$$\int_{-\infty}^{\infty} e^{\gamma_1 t} \widetilde{\phi}(t) dt < \infty. \quad (3.20)$$

Then by (3.12), (3.20), and the same argument as the proof of Lemma 3.3 of [DKS] we have

$$|w_1(s)| = -M_0 A_1(\beta) I_1 e^{-\gamma_1 s} (1 + o(1)) \quad \text{as } s \rightarrow \infty$$

where

$$I_1 = \int_{-\infty}^{\infty} e^{\gamma_1 t} \widetilde{\phi}(t) dt$$

and the theorem follows. □

Corollary 3.1. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha = \frac{2\beta+\rho_1}{1-m}$ and $\beta > \beta_1$. Let v_{λ_1} , v_{λ_2} , be two radially symmetric solution of (1.3) in \mathbb{R}^n with $v_{\lambda_1}(0) = \lambda_1$, $v_{\lambda_2}(0) = \lambda_2$ which satisfies (1.10) with $\lambda = \lambda_1, \lambda_2$, and $B = B_{\lambda_1}, B_{\lambda_2}$, respectively. Then*

$$B_{\lambda_2} = (\lambda_1/\lambda_2)^{\frac{(1-m)\gamma_1}{2}} B_{\lambda_1}. \quad (3.21)$$

Proof: By Corollary 2.7, (2.26) holds. By Theorem 1.3,

$$v_{\lambda_i}(x) = \left(\frac{C_*}{|x|^2} \right)^{\frac{1}{1-m}} (1 - B_{\lambda_i} |x|^{-\gamma_1} + o(|x|^{-\gamma_1})) \quad \forall i = 1, 2 \quad \text{as } |x| \rightarrow \infty. \quad (3.22)$$

Hence by (2.26) and (3.22),

$$\begin{aligned} v_{\lambda_2}(x) &= \frac{\lambda_2}{\lambda_1} \left(\frac{C_*}{((\lambda_2/\lambda_1)^{\frac{1-m}{2}} |x|)^2} \right)^{\frac{1}{1-m}} (1 - B_{\lambda_1} ((\lambda_2/\lambda_1)^{\frac{1-m}{2}} |x|)^{-\gamma_1} + o(|x|^{-\gamma_1})) \quad \text{as } |x| \rightarrow \infty \\ &= \left(\frac{C_*}{|x|^2} \right)^{\frac{1}{1-m}} (1 - B_{\lambda_1} ((\lambda_2/\lambda_1)^{\frac{1-m}{2}} |x|)^{-\gamma_1} + o(|x|^{-\gamma_1})) \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (3.23)$$

By (3.22) and (3.23), we get (3.21) and the corollary follows. \square

4 Second order asymptotic of blow-up solutions

In this section we will prove Theorem 1.4.

Proof of Theorem 1.4: Similar to section 3 we let

$$w(s) = [(r^2/C_*)^{\frac{1}{1-m}} g_\lambda(r)]^m - 1, \quad r = e^s, \quad s \in \mathbb{R}.$$

Then w satisfies (3.4) in \mathbb{R} . By the variation of parameter formula for any $s_0 \in \mathbb{R}$ there exist constants $C_2(s_0), C_3(s_0)$, such that

$$w(s) = \frac{1}{\gamma_2 - \gamma_1} \left(e^{-\gamma_1 s} \int_{s_0}^s e^{\gamma_1 t} f(t) dt - e^{-\gamma_2 s} \int_{s_0}^s e^{\gamma_2 t} f(t) dt \right) + C_2(s_0) e^{-\gamma_1 s} - C_3(s_0) e^{-\gamma_2 s} \quad (4.1)$$

holds for any $s \in \mathbb{R}$. By Lemma 2.4, (2.7) holds. Hence $w(s) > 0$ for all $s \in \mathbb{R}$. By (2.27), $w(s) \rightarrow 0$ as $s \rightarrow \infty$. By (4.1) and integration by parts,

$$w(s) = -M_0 \left\{ A_1(\beta) e^{-\gamma_1 s} \int_{s_0}^s e^{\gamma_1 t} \tilde{\phi}(t) dt - A_2(\beta) e^{-\gamma_2 s} \int_{s_0}^s e^{\gamma_2 t} \tilde{\phi}(t) dt \right\} + C'_2(s_0) e^{-\gamma_1 s} - C'_3(s_0) e^{-\gamma_2 s} \quad (4.2)$$

for any $s_0, s \in \mathbb{R}$ where M_0 and $A_1(\beta), A_2(\beta)$, are given by (3.13) and (3.14) respectively and

$$\begin{cases} C'_2(s_0) = C_2(s_0) + \frac{M_0 \beta}{\rho_1} \tilde{\phi}(s_0) e^{\gamma_1 s_0} \\ C'_3(s_0) = C_3(s_0) + \frac{M_0 \beta}{\rho_1} \tilde{\phi}(s_0) e^{\gamma_2 s_0}. \end{cases}$$

Let $\beta_2 > 0$ be as in the proof of Theorem 1.3 and $\beta > \beta_2$. Then by the proof of Theorem 1.3,

$$A_1(\beta) > 0 > A_2(\beta) \quad (4.3)$$

$$\Leftrightarrow \gamma_1 < \frac{\rho_1}{(1-m)\beta} < \gamma_2. \quad (4.4)$$

Since $w(s) \rightarrow \infty$ as $r = e^s \rightarrow 0^+$,

$$\begin{aligned}\widetilde{\phi}(s) &\approx (1 + w(s))^{\frac{1}{m}} \approx w(s)^{\frac{1}{m}} \approx \frac{r^{\frac{2}{1-m}} g_\lambda(r)}{C_*^{\frac{1}{1-m}}} \approx \frac{(\lambda e^s)^{-\frac{\rho_1}{(1-m)\beta}}}{C_*^{\frac{1}{1-m}}} \quad \text{as } r = e^s \rightarrow 0^+ \\ \Rightarrow \lim_{s \rightarrow -\infty} \widetilde{\phi}(s) e^{\frac{\rho_1 s}{(1-m)\beta}} &= \lim_{s \rightarrow -\infty} w(s)^{\frac{1}{m}} e^{\frac{\rho_1 s}{(1-m)\beta}} = \lambda^{-\frac{\rho_1}{(1-m)\beta}} C_*^{-\frac{1}{1-m}},\end{aligned}\tag{4.5}$$

multiplying (4.2) by $e^{\gamma_2 s}$ and letting $s \rightarrow -\infty$, by (4.3), (4.4) and (4.5) we get,

$$\begin{aligned}& -M_0 \left\{ A_1(\beta) \lim_{s \rightarrow -\infty} e^{(\gamma_2 - \gamma_1)s} \int_{s_0}^s e^{\gamma_1 t} \widetilde{\phi}(t) dt - |A_2(\beta)| \int_{-\infty}^{s_0} e^{\gamma_2 t} \widetilde{\phi}(t) dt \right\} - C'_3(s_0) \\ &= \lim_{s \rightarrow -\infty} e^{(\gamma_2 - \frac{m\rho_1}{(1-m)\beta})s} \cdot \lim_{s \rightarrow -\infty} e^{\frac{m\rho_1 s}{(1-m)\beta}} w(s) \\ &= 0.\end{aligned}\tag{4.6}$$

By (4.5) there exist constants $C_4 > 0$, $C_5 > 0$ and $s_1 < 0$ such that

$$C_4 \leq \widetilde{\phi}(s) e^{\frac{\rho_1 s}{(1-m)\beta}} \leq C_5 \quad \forall s \leq s_1.\tag{4.7}$$

Then by (4.4) and (4.7),

$$\begin{aligned}\left| \int_{s_0}^s e^{\gamma_1 t} \widetilde{\phi}(t) dt \right| &\geq \frac{C_4 \left\{ e^{(\gamma_1 - \frac{\rho_1}{(1-m)\beta})s} - e^{(\gamma_1 - \frac{\rho_1}{(1-m)\beta})s_1} \right\}}{\frac{\rho_1}{(1-m)\beta} - \gamma_1} - \left| \int_{s_0}^{s_1} e^{\gamma_1 t} \widetilde{\phi}(t) dt \right| \quad \forall s < s_1 \\ &\rightarrow \infty \quad \text{as } s \rightarrow -\infty.\end{aligned}\tag{4.8}$$

Hence by (4.4), (4.5), (4.8) and the l'Hospital rule,

$$\lim_{s \rightarrow -\infty} \frac{\int_{s_0}^s e^{\gamma_1 t} \widetilde{\phi}(t) dt}{e^{(\gamma_1 - \gamma_2)s}} = \frac{1}{(\gamma_1 - \gamma_2)} \cdot \lim_{s \rightarrow -\infty} e^{\gamma_2 s} \widetilde{\phi}(s) = \frac{1}{(\gamma_1 - \gamma_2)} \lim_{s \rightarrow -\infty} e^{(\gamma_2 - \frac{\rho_1}{(1-m)\beta})s} \cdot \lim_{s \rightarrow -\infty} e^{\frac{\rho_1 s}{(1-m)\beta}} \widetilde{\phi}(s) = 0.\tag{4.9}$$

By (4.6) and (4.9),

$$C'_3(s_0) = M_0 |A_2(\beta)| \int_{-\infty}^{s_0} e^{\gamma_2 t} \widetilde{\phi}(t) dt > 0 \quad \forall s_0 \in \mathbb{R}.\tag{4.10}$$

Putting $s = s_0$ in (4.2),

$$C'_2(s_0) = e^{\gamma_1 s_0} w(s_0) + e^{(\gamma_1 - \gamma_2)s_0} C'_3(s_0) > 0 \quad \forall s_0 \in \mathbb{R}.\tag{4.11}$$

By (4.2), (4.3) and (4.10),

$$0 < w(s) \leq C'_2(s_0) e^{-\gamma_1 s} \quad \forall s > s_0.\tag{4.12}$$

Since $w(s) \rightarrow 0$ as $s \rightarrow \infty$, there exists $s_1 > 0$ such that

$$0 \leq \widetilde{\phi}(s) \leq a_2 w(s)^2 \quad \forall s > s_1\tag{4.13}$$

where the constant $a_2 > 0$ is as given in (3.5). By (4.12) and (4.13),

$$0 \leq \widetilde{\phi}(s) \leq C_6 e^{-2\gamma_1 s} \quad \forall s > s_1 \quad (4.14)$$

where $C_6 = a_2 C'_2(0)^2$. Multiplying (4.2) by $e^{\gamma_1 s}$ and letting $s \rightarrow \infty$, by (4.3), (4.4) and (4.5),

$$\begin{aligned} & \lim_{s \rightarrow \infty} e^{\gamma_1 s} w(s) \\ &= -M_0 \left\{ A_1(\beta) \int_{s_0}^{\infty} e^{\gamma_1 t} \widetilde{\phi}(t) dt + |A_2(\beta)| \lim_{s \rightarrow \infty} e^{(\gamma_1 - \gamma_2)s} \int_{s_0}^s e^{\gamma_2 t} \widetilde{\phi}(t) dt \right\} + C'_2(s_0) \quad \forall s_0 \in \mathbb{R}. \end{aligned} \quad (4.15)$$

By (4.14),

$$\begin{aligned} & 0 \leq \int_{s_0}^s e^{\gamma_2 t} \widetilde{\phi}(t) dt \leq C_7(1 + e^{(\gamma_2 - 2\gamma_1)s}) \quad \forall s > \max(s_0, s_1), s_0 \in \mathbb{R} \\ \Rightarrow & 0 \leq e^{(\gamma_1 - \gamma_2)s} \int_{s_0}^s e^{\gamma_2 t} \widetilde{\phi}(t) dt \leq C_7(e^{(\gamma_1 - \gamma_2)s} + e^{-\gamma_1 s}) \quad \forall s > \max(s_0, s_1), s_0 \in \mathbb{R} \\ \Rightarrow & \lim_{s \rightarrow \infty} e^{(\gamma_1 - \gamma_2)s} \int_{s_0}^s e^{\gamma_2 t} \widetilde{\phi}(t) dt = 0 \quad \forall s_0 \in \mathbb{R} \end{aligned} \quad (4.16)$$

where $C_7 > 0$ is some constant depending on s_0 and s_1 . By (4.10), (4.11), (4.15) and (4.16), the limit

$$B := \lim_{s \rightarrow -\infty} e^{\gamma_1 s} w(s)$$

exists and is given by

$$0 \leq B = M_0 |A_2(\beta)| e^{(\gamma_1 - \gamma_2)s} \int_{-\infty}^s e^{\gamma_2 t} \widetilde{\phi}(t) dt - M_0 A_1(\beta) \int_s^{\infty} e^{\gamma_1 t} \widetilde{\phi}(t) dt + e^{\gamma_1 s} w(s) \quad \forall s \in \mathbb{R}. \quad (4.17)$$

We claim that $B > 0$. Suppose not. Then $B = 0$. Hence by (4.17),

$$w(s) \leq M_0 A_1(\beta) e^{-\gamma_1 s} \int_s^{\infty} e^{\gamma_1 t} \widetilde{\phi}(t) dt \quad \forall s \in \mathbb{R}. \quad (4.18)$$

We now choose $\varepsilon \in (0, \frac{\gamma_1}{a_2 M_0 A_1(\beta)})$ where a_2 is as given in (3.5) and we choose $s_2 > s_1$ such that

$$e^{-\gamma_1 s_2} \leq \varepsilon \quad \text{and} \quad e^{\gamma_1 s} w(s) \leq 1 \quad \forall s \geq s_2. \quad (4.19)$$

Let $a_3 = \frac{a_2 M_0 A_1(\beta) \varepsilon}{\gamma_1}$. Then $0 < a_3 < 1$. By (4.13), (4.18) and (4.19),

$$w(s) \leq a_2 M_0 A_1(\beta) e^{-\gamma_1 s} \int_s^{\infty} e^{-\gamma_1 t} dt \leq \frac{a_2 M_0 A_1(\beta)}{\gamma_1} e^{-2\gamma_1 s} \leq a_3 e^{-\gamma_1 s} \quad \forall s \geq s_2. \quad (4.20)$$

By (4.13), (4.18), (4.19) and (4.20),

$$w(s) \leq a_2 M_0 A_1(\beta) a_3^2 e^{-\gamma_1 s} \int_s^{\infty} e^{-\gamma_1 t} dt \leq \frac{a_2 M_0 A_1(\beta)}{\gamma_1} a_3^2 e^{-2\gamma_1 s} \leq a_3^3 e^{-\gamma_1 s} \quad \forall s \geq s_2. \quad (4.21)$$

Repeating the above argument we get that

$$\begin{aligned} w(s) &\leq a_3^{2^{k+1}-1} \cdot e^{-\gamma_1 s} \quad \forall s \geq s_2, k \in \mathbb{Z}^+ \\ \Rightarrow w(s) &\equiv 0 \quad \forall s \geq s_2 \quad \text{as } k \rightarrow \infty \end{aligned}$$

which contradicts the fact that $w(s) > 0$ for all $s \in \mathbb{R}$. Hence $B > 0$ and the theorem follows. \square

Note that when $m = \frac{n-2}{n+2}$, $n \geq 3$, $\rho_1 = 1$, $\beta > \beta_1 = \frac{1}{2m}$, then by the result of [DKS] (4.3) holds. Moreover

$$\begin{aligned} \gamma_1 - \frac{1}{(1-m)\beta} &= \frac{\beta(n-2) - \sqrt{\beta^2(n-2)^2 - 4(n-2)}}{2} - \frac{1}{(1-m)\beta} \\ &= \frac{2(n-2)}{\beta(n-2) + \sqrt{\beta^2(n-2)^2 - 4(n-2)}} - \frac{n+2}{4\beta} \\ &= -\frac{(n+2)\sqrt{\beta^2(n-2)^2 - 4(n-2)} + (n-2)(n-6)\beta}{4\beta(\beta(n-2) + \sqrt{\beta^2(n-2)^2 - 4(n-2)})} \\ &< 0 \quad \text{if } \beta > \frac{1}{2m} \end{aligned}$$

and

$$\begin{aligned} \gamma_2 - \frac{1}{(1-m)\beta} &= \frac{\beta(n-2) + \sqrt{\beta^2(n-2)^2 - 4(n-2)}}{2} - \frac{n+2}{4\beta} \\ &= \frac{2\beta^2(n-2) + 2\beta\sqrt{\beta^2(n-2)^2 - 4(n-2)} - (n+2)}{4\beta} \\ &> 0 \end{aligned}$$

if

$$4\beta^2[\beta^2(n-2)^2 - 4(n-2)] > [n+2 - 2\beta^2(n-2)]^2 \Leftrightarrow \beta > \beta_1 = \frac{1}{2m}.$$

Hence (4.4) holds when $m = \frac{n-2}{n+2}$, $n \geq 3$, $\rho_1 = 1$, $\beta > \beta_1 = \frac{1}{2m}$. Then our proof above gives another proof of the following result of [DKS].

Theorem 4.1. *Let $n \geq 3$, $m = \frac{n-2}{n+2}$, $\rho_1 = 1$, $\lambda > 0$, $\beta > \beta_1$, $\alpha = \frac{2\beta+1}{1-m}$. If g_λ is a radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ which satisfies (1.12), then (1.16) holds for some constants $B > 0$ where γ_1 is given by (1.8).*

By Corollary 2.6 and an argument similar to the proof of Corollary 3.1 we have the following result.

Corollary 4.2. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha = \frac{2\beta+\rho_1}{1-m}$ and $\beta > \beta_1$. Suppose g_{λ_1} , g_{λ_2} are two radially symmetric solution of (1.3) in $\mathbb{R}^n \setminus \{0\}$ which satisfies (1.12) and (1.16) with $\lambda = \lambda_1, \lambda_2$, and $B = B_1, B_2$, respectively. Then (3.21) holds.*

5 Large time behaviour of solutions

In this section we will prove Theorem 1.6 and Theorem 1.7. Since the proof of Theorem 1.6 and Theorem 1.7 are similar to the proof of [DS2] and [HuK], we will only sketch its proof here. We first observe that by an argument similar to the proof of Corollary 2.2 of [DS1] we have the following two lemmas.

Lemma 5.1. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$ and let u_1, u_2 be two solutions of (1.2) in $\mathbb{R}^n \times (0, T)$ with initial values $u_{0,1} \geq 0, u_{0,2} \geq 0$, respectively. Suppose $u_{0,1} - u_{0,2} \in L^1(\mathbb{R}^n)$ and for any $0 < T_1 < T$ there exist constants $r_0 > 0, C > 0$, such that either $u_1(x, t) \geq C/|x|^{\frac{2}{1-m}}$ for all $|x| \geq r_0, 0 < t < T_1$, or $u_2(x, t) \geq C/|x|^{\frac{2}{1-m}}$ for all $|x| \geq r_0, 0 < t < T_1$ holds. Then*

$$\int_{\mathbb{R}^n} |u_1(x, t) - u_2(x, t)| dx \leq \int_{\mathbb{R}^n} |u_{0,1} - u_{0,2}| dx \quad \forall 0 < t < T.$$

Hence if \tilde{u}_1, \tilde{u}_2 , are the rescaled solution of u_1, u_2 , given by (1.17),

$$\int_{\mathbb{R}^n} |\tilde{u}_1(x, s) - \tilde{u}_2(x, s)| dx \leq e^{-(n\beta-\alpha)s} \int_{\mathbb{R}^n} |u_{0,1} - u_{0,2}| dx \quad \forall s > -\log T.$$

Lemma 5.2. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\beta > \frac{m\rho_1}{n-2-nm}$, $\alpha = \frac{2\beta+\rho_1}{1-m}$, and let u_1, u_2 be two solutions of (1.2) in $(\mathbb{R}^n \setminus \{0\}) \times (0, T)$ with initial values $u_{0,1} \geq 0, u_{0,2} \geq 0$, respectively and there exist constants $C_1 > 0, C_2 > 0$, such that*

$$C_1 \leq |x|^{\frac{\alpha}{\beta}} u_i(x) \leq C_2 \quad \forall 0 < |x| \leq 1, x \in \mathbb{R}^n, i = 1, 2. \quad (5.1)$$

Suppose $u_{0,1} - u_{0,2} \in L^1(\mathbb{R}^n \setminus \{0\})$ and for any $0 < T_1 < T$ there exist constants $r_0 > 0, C > 0$, such that either $u_1(x, t) \geq C/|x|^{\frac{2}{1-m}}$ for all $|x| \geq r_0, 0 < t < T_1$, or $u_2(x, t) \geq C/|x|^{\frac{2}{1-m}}$ for all $|x| \geq r_0, 0 < t < T_1$ holds. Then

$$\int_{\mathbb{R}^n \setminus \{0\}} |u_1(x, t) - u_2(x, t)| dx \leq \int_{\mathbb{R}^n \setminus \{0\}} |u_{0,1} - u_{0,2}| dx \quad \forall 0 < t < T.$$

Hence if \tilde{u}_1, \tilde{u}_2 , are the rescaled solution of u_1, u_2 , given by (1.17) and $\beta > \beta_1$, then

$$\int_{\mathbb{R}^n \setminus \{0\}} |\tilde{u}_1(x, s) - \tilde{u}_2(x, s)| dx \leq e^{-(n\beta-\alpha)s} \int_{\mathbb{R}^n \setminus \{0\}} |u_{0,1} - u_{0,2}| dx \quad \forall s > -\log T.$$

By (1.4), (2.27), Lemma 5.1, Lemma 5.2 and the same argument as the proof of Theorem 1.1 of [HuK] we get Theorem 1.7 and the following result.

Theorem 5.3. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $T > 0$, $\rho_1 = 1$, $\beta > \beta_1$ and $\alpha = \frac{2\beta+1}{1-m}$. Let ψ_λ be given by (1.5) and let u_0 satisfy*

$$\psi_{\lambda_1}(x, 0) \leq u_0(x) \leq \psi_{\lambda_2}(x, 0) \quad \text{in } \mathbb{R}^n$$

and

$$u_0(x) - \psi_{\lambda_0}(x, 0) \in L^1(\mathbb{R}^n) \quad (5.2)$$

for some constants $\lambda_2 > \lambda_1 > 0$ and $\lambda_0 > 0$. Let u be the maximal solution of (1.2) and \tilde{u} be given by (1.17). Then the rescaled solution $\tilde{u}(\cdot, s)$ converges uniformly on every compact subset of \mathbb{R}^n and in $L^1(\mathbb{R}^n)$ to v_{λ_0} as $s \rightarrow \infty$. Moreover,

$$\|\tilde{u}(\cdot, s) - v_{\lambda_0}\|_{L^1(\mathbb{R}^n)} \leq e^{-(n\beta-\alpha)s} \|u_0 - \psi_{\lambda_0}(\cdot, 0)\|_{L^1(\mathbb{R}^n)} \quad \forall s > -\log T.$$

By Theorem 5.3 and an argument similar to the proof of Theorem 1.2 of [HuK], Theorem 1.6 follows.

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